Expected Utility Preferences for Contingent Claims and Lotteries

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Abstract

In Arrow's seminal analysis of optimal risk bearing in which he introduced contingent claim securities, he assumed preferences were representable by a state independent Expected Utility function. Although the classic contingent claim setting assumes agents choose over contingent consumption vectors conditioned on a fixed set of probabilities, later work on information economics suggested that allowing probabilities to change across contingent claim spaces could be an interesting extension. However the set of axioms that are necessary and sufficient for the existence of an Expected Utility representation for the classic contingent claim space with a fixed set of probabilities does not ensure that this form utility extends across multiple contingent claim spaces. In this paper, we derive a set of axioms on preferences which are necessary and sufficient for the existence of an Expected Utility representation when probabilities change. We also consider the incremental axioms which are necessary and sufficient for Expected Utility preferences to extend to the classic lottery setting of von Neumann and Morgenstern, where agents choose not only over consumption vectors but also over probabilities vectors.

KEYWORDS. Expected utility, contingent claim demand, additive separability, lottery preferences, contingent claim preferences

JEL CLASSIFICATION. D01, D11, D80.

1 Introduction

In the classic Arrow-Debreu contingent claim set up, one typically assumes that there are a finite number of states and agents possess preferences over state contingent consumption. Arrow (1953) introduced contingent claim securities and derived conditions such that the allocation of risk-bearing by competitive securities markets is optimal. To simplify his analysis and derive clear results, he assumed that preferences are representable by a state independent von Neumann-Morgenstern Expected Utility function. In Arrow's analysis, probabilities are fixed. Over the ensuing two decades, researchers began to consider the case of changing probabilities as they explored questions of speculation and the acquisition and value of information (see Rubinstein 1975, Hirshleifer and Riley 1979 for a classic overview and Schlee 2001 for more a more recent example). This literature continued to assume that risk preferences are representable by a state independent Expected Utility function as new information is obtained and probabilities vary.

Arrow, in explicitly citing the von Neumann-Morgenstern theorem, effectively borrowed their result despite the fact that his state contingent claim setting is quite different from the lottery setting assumed by von Neumann-Morgenstern (1953) and Samuelson (1952). The contingent claim setting is based on a fixed set of state probabilities and varying state consumption payoffs, whereas the lottery setting assumes arbitrary probability distributions where both probabilities and consumption payoffs can vary. Clearly the lottery setting is more general, but the assumed completeness axiom requiring individuals to have preferences over the full space of distributions is very strong. Indeed both von Neumann and Morgenstern (1953, p. 630) and Aumann (1962) argue that the completeness axiom is of "dubious validity".

Thus, it would seem desirable to develop an alternative set of axioms tailored to the narrower contingent claim choice space. This new set of axioms needs to accommodate the fact that the latter space, unlike the former, is not a mixture space.[1] Werner (2005) (building on Hens 1992) derived such a set of axioms which

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[1]For example since the set of contingent claim distributions is not a mixture space, one can not assume the Strong Independence Axiom which is central to the von Neumann-Morgenstern

are necessary and sufficient for preferences to be representable by a state independent Expected Utility function. He followed the classic contingent claim set up in assuming a fixed set of state probabilities. However in applications such as the information models mentioned above and the contingent claim demand analysis in Kubler, Selden and Wei (2014), it is natural to suppose that probabilities can vary. In this case, preferences need to be viewed as being defined on a set of contingent claim spaces characterized by a set of state probabilities. Unfortunately assuming Werner's axioms hold on each contingent claim space does not ensure that the NM index of the consumer's state independent Expected Utility function will be the same across the different spaces as probabilities change.

In this paper, we provide the additional axiomatic structure which is necessary and sufficient to extend Werner's analysis to the case where state probabilities can vary and thereby avoid making the overly strong assumption that agents possess preferences over the space of all possible probability distributions. One key axiom we use to ensure that the NM index is unchanged (up to a positive affine transform) across different contingent claim spaces is a modified version of Tradeoff Consistency introduced by Wakker (1989) in a SEU (Subjective Expected Utility) setting. We require a modified version of this axiom, because for us probabilities are exogenous, and not endogenous as in the SEU case.

One key aspect of preferences defined over the full set of contingent claim spaces corresponding to different probability vectors, is that the consumer chooses over consumption vectors, but never gets to choose over probabilities. This is in contrast to the case where preferences defined over risk prospects or lotteries and the decision maker can be viewed as choosing over both vectors of consumption plans and probabilities. We provide the incremental axioms which are necessary and sufficient to extend Expected Utility preferences with a non-changing NM index to the space of risky prospects. For this case, a Certainty Uniqueness axiom must be added. Alternatively, one can use a variation of the Werner axioms and an axiom similar to Probabilistic Sophistication (introduced by Machina and Schmeidler 1992 in a SEU setting).

The rest of the paper is organized as follows. In the next section, we compare and contrast the choice spaces and Expected Utility representations associated with (i) contingent claims assuming a fixed set of probabilities, (ii) contingent claims assuming state probabilities vary as parameters and (iii) a set of probability distributions or risky prospects corresponding to the case where both probabilities and consumption vectors are choice variables. Section 3 develops the axiom

(1953) and Samuelson (1952) Expected Utility theorem. In the contingent claim setting, Expected Utility must be based on an alternative set of axioms.

system for Expected Utility defined over contingent claims. First we review the axioms for each contingent claim space conditioned on a fixed set of probabilities. Then we consider the set of axioms across different contingent claim spaces with varying probabilities, which result in an Expected Utility representation which is (i) characterized by the NM index being independent of probabilities and (ii) consistent with the demand tests discussed in Kubler Selden and Wei (2014). In Section 4, we identify the incremental set of axioms required to go from Expected Utility preferences defined over a set of contingent claim spaces to Expected Utility preferences defined over the space of distribution, where the number of states is finite. Section 5 gives an example illustrating how a utility transformation dependent on probabilities, which does not affect contingent claim demands, can radically alter the shape of indifference curves in the Marschak-Machina probability triangle. An Appendix is provided in which the indirect utility function used for the Section 5 Example 4 is derived.

2 Different Preference Domains

Assume there are S states of nature and there is a single consumption good in each state. A typical consumption plan is an S vector $(c_1, c_2, ..., c_S)$ in the consumption space defined by $\mathbb{R}_+^S$. We assume that probabilities are objective and known and denote the probability of state s by π_s. Let $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_S)$, where $\boldsymbol{\pi} \in \Delta^{S-1} = \{\boldsymbol{\pi} \in \mathbb{R}_+^S : \sum_{s=1}^{S} \pi_s = 1\}$. Given this setting, we next define three different choices spaces which we will investigate.

The first preference domain we consider corresponds to the classic Arrow-Debreu contingent claim setup in which for each value of $\boldsymbol{\pi} \in \Delta^{S-1}$ a decision maker is assumed to have complete, transitive and continuous preferences over $\mathbb{R}_+^S$ which is denoted $\preceq_{\boldsymbol{\pi}}$. The second preference domain arises if one assumes as in Kubler, Selden and Wei (2014) that the consumer confronts a sequence of independent contingent claim optimizations where probabilities and prices vary. Then corresponding to a set of probability vectors $\{\boldsymbol{\pi}\}$, there will be a set of preference relations $\{\preceq_{\boldsymbol{\pi}}\}$ which need not be equivalent. The set of preference orderings is assumed to be representable by a continuous and strictly increasing utility function $U(\mathbf{c}; \boldsymbol{\pi}) : \mathbb{R}_+^S \to \mathbb{R}$, which is C^1 in $\boldsymbol{\pi}$ and where the notation $U(\mathbf{c}; \boldsymbol{\pi})$ indicates that corresponding to each $\boldsymbol{\pi}$, there will be a potentially different utility. It should be emphasized that for this set of utilities, the probability vector $\boldsymbol{\pi}$ is allowed to change, but only as a parameter. One can view $U(\mathbf{c}; \boldsymbol{\pi})$ as being defined over a series of contingent claim spaces but not on their union. Therefore, although we can use $U(\mathbf{c}; \boldsymbol{\pi})$ to compare lotteries in each given contingent claim

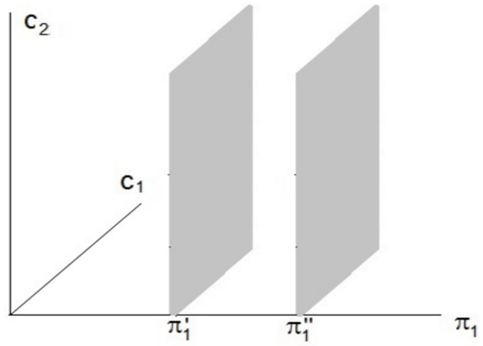


Figure 1:

space, it cannot be used to compare the lotteries across the different contingent claim spaces. This is expressed geometrically in Figure 1, where two states are assumed. Each shaded plane in the figure corresponds to a contingent claim space with a given π_1. Preferences on the planes corresponding to π_1' and π_1'' are represented respectively by $U(\mathbf{c};\boldsymbol{\pi}')$ and $U(\mathbf{c};\boldsymbol{\pi}'')$.

The third choice space we consider is the full set of distributions corresponding to $(\mathbf{c},\boldsymbol{\pi})$, or the set of "risky prospects". To make this precise, define a risky prospect as a pair of vectors $(\mathbf{c},\boldsymbol{\pi}) \in \mathbb{R}_+^S \times \Delta^{S-1}$. Assume that a decision maker has continuous, complete and transitive preferences over $\mathcal{P} = \mathbb{R}_+^S \times \Delta^{S-1}$, denoted $\preceq_{\mathcal{P}}$. For any fixed $\boldsymbol{\pi} \in \Delta^{S-1}$ this implies preferences $\preceq_{\boldsymbol{\pi}}$ are well defined. To distinguish the representation of $\preceq_{\mathcal{P}}$ from the representation of $\{\preceq_{\boldsymbol{\pi}}\}$, we use the notation $U(\mathbf{c},\boldsymbol{\pi})$ instead of $U(\mathbf{c};\boldsymbol{\pi})$. The former, in contrast to the latter, has both $\mathbf{c}$ and $\boldsymbol{\pi}$ as arguments since one can compare lotteries across different contingent claim spaces, or slices in Figure 1.

For each of the above three preferences cases, we provide in the next two Sections a set of axioms that is necessary and sufficient for preferences to be representable by an Expected Utility function. We next illustrate the difference in the resulting Expected Utilities using the following example[2]

$$U(c_1,c_2,c_3;\pi_1,\pi_2,\pi_3) = -\pi_1 \sum_{s=1}^{3} \pi_s \left(\exp(-\pi_1 c_s) + \exp(-\pi_2 c_s) + \exp(-\pi_3 c_s)\right). \tag{1}$$

Note first that if, as in the standard contingent claim case, probabilities are fixed at $\pi_1 = 0.5$, $\pi_2 = 0.3$ and $\pi_3 = 0.2$ (defining a specific slice in Figure 1), eqn. (1)

[2] This utility will be recognized to be a modified version of a representation discussed in Kubler, Selden and Wei (2014).

is equivalent up to a positive affine transformation to

$$\begin{aligned} U(\mathbf{c};\boldsymbol{\pi}) &= -0.5\left(\exp(-0.5c_1)+\exp(-0.3c_1)+\exp(-0.2c_1)\right) \\ &\quad -0.3\left(\exp(-0.5c_2)+\exp(-0.3c_2)+\exp(-0.2c_2)\right) \\ &\quad -0.2\left(\exp(-0.5c_3)+\exp(-0.3c_3)+\exp(-0.2c_3)\right). \end{aligned} \tag{2}$$

Moreover it can be verified that

$$\left.\frac{\partial U/\partial c_1}{\partial U/\partial c_s}\right|_{c_1=c_s} = \frac{\pi_1}{\pi_s} \quad (s=2,3) \tag{3}$$

and the utility (2) passes the Expected Utility test in Dybvig (1983), implying that it can be viewed as an Expected Utility when probabilities are fixed and the NM index is given by

$$v(c) = -\left(\exp(-0.5c)+\exp(-0.3c)+\exp(-0.2c)\right). \tag{4}$$

However when probabilities are allowed to vary and one considers preferences on different contingent claim spaces, the resulting contingent claim demands cannot pass the tests discussed in Kubler, Selden and Wei (2014). The reason is that when probabilities vary, the NM index associated with the utility (1) will also change. In general, the kind of utility function in (1) takes the form

$$U(\mathbf{c};\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S}\pi_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{5}$$

where f is increasing in its second argument and the NM index $v_{\boldsymbol{\pi}}$ is allowed to depend on probabilities. It should be emphasized that for the utility (1), the NM index

$$v_{\boldsymbol{\pi}}(c) = \exp(-\pi_1 c)+\exp(-\pi_2 c)+\exp(-\pi_3 c) \tag{6}$$

depends on $\boldsymbol{\pi}$ but is state independent and thus is not denoted by $v_{s,\boldsymbol{\pi}}$. The notation $f(\boldsymbol{\pi},\cdot)$ indicates that on each contingent claim slice corresponding to each probability vector $\boldsymbol{\pi}$, one can consider a different increasing monotonic transform of the Expected Utility $\sum_{s=1}^{S}\pi_s v(c_s)$ and optimal contingent claim demands will not be altered.

Next consider the utility function

$$U(c_1,c_2,c_3;\pi_1,\pi_2,\pi_3) = -\pi_1\sum_{s=1}^{3}\pi_s\left(\exp(-0.5c_s)+\exp(-0.3c_s)+\exp(-0.2c_s)\right). \tag{7}$$

If one ignores the π_1 in front, this is a standard Expected Utility with the same NM index on each contingent claim slice

$$v(c) = -\left(\exp(-0.5c)+\exp(-0.3c)+\exp(-0.2c)\right). \tag{8}$$

More generally, the utility (7) takes the form

$$U(\mathbf{c};\boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{9}$$

where f continues to be increasing in its second argument but the NM index v is independent of probabilities $\boldsymbol{\pi}$. Since (7) is an Expected Utility on each contingent claim slice in Figure 1 and the NM index is the same on each slice, it will result in demands that pass the tests in Kubler, Selden and Wei (2014). From observing optimal contingent claim demands, one can never distinguish ordinal transformations in the utility function corresponding to $f(\boldsymbol{\pi}, \cdot)$. However when considering comparisons over lotteries, the utility function defined in (7) (and more generally (9)) cannot be viewed as an Expected Utility function. To see this, consider the following two lotteries

$$L_1 =< 1, 2, 3; 0.2, 0.3, 0.5 > \text{ and } L_2 =< 2, 1, 3; 0.3, 0.2, 0.5 >, \tag{10}$$

where the payoffs in L_1 and L_2, respectively, are given by $1, 2, 3$ and $2, 1, 3$ and the probabilities by $0.2, 0.3, 0.5$ and $0.3, 0.2, 0.5$. Clearly for any Expected Utility maximizer, L_1 and L_2 will be indifferent. However for the utility function (7) since $\pi_1 = 0.2$ for L_1 and $\pi_1 = 0.3$ for L_2, we have

$$U(L_1) < U(L_2). \tag{11}$$

Hence from the lottery point of view, the transformation $f(\boldsymbol{\pi}, x) = \pi_1 x$ affects the consumer's choice whereas it does not in a demand optimization. Because of the transformation, the probabilities do not enter into the utility function linearly and (7) is not an Expected Utility function. The probability weighting function for state i ($i = 1, 2, 3$) is $\pi_1\pi_i$. From this perspective, this utility form can be viewed as being more analogous to a Prospect Theory form (see Kahneman and Tversky 1979) than Expected Utility.

Finally for the third choice space where preferences over lotteries (for a finite number of states S), $\preceq_{\mathcal{P}}$, are represented by an Expected Utility function, the representation will take the form

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{12}$$

where f is independent of probabilities and increasing and the NM index v is independent of probabilities. For instance in terms of the examples considered above, $U(\mathbf{c}, \boldsymbol{\pi})$ can take any monotone transform of

$$-\sum_{s=1}^{3} \pi_s \left(\exp\left(-0.5c_s\right) + \exp\left(-0.3c_s\right) + \exp\left(-0.2c_s\right)\right). \tag{13}$$

Remark 1 *The negative exponential utility function (1) is used in this section to illustrates the different Expected Utility functions and respective preference domains considered in this paper. However the reader may not find the dependence of the NM index on probabilities particularly intuitive. A perhaps more behaviorally plausible example is given by the following*

$$U(\mathbf{c};\boldsymbol{\pi}) = \pi_1 c_1^{\frac{\pi_1}{2}} + \pi_2 c_2^{\frac{\pi_1}{2}} + \pi_3 c_3^{\frac{\pi_1}{2}}. \tag{14}$$

Without loss of generality, we can always assume that the state $s = 1$ is associated with the maximum consumption payoff. In this case it would seem reasonable that increasing the probability π_1 of this best state would result in the consumer becoming less risk averse. Computing the Arrow-Pratt measure of relative risk aversion for the NM index, one obtains $-cu''(c)/u'(c) = 1 - \frac{\pi_1}{2}$. This is consistent with the intuition – increasing π_1 results in a decrease of the Arrow-Pratt measure. The contingent claim demands, given a fixed set of probabilities, generated by the utility (14) will satisfy the Expected Utility test of Dybvig (1983), but across contingent claim slices associated with varying probabilities the demands will not pass the demand tests of Kubler, Selden and Wei (2014).

3 Preferences over Contingent Claims

In this section, we derive Expected Utility representations assuming preferences are defined over a single or set of contingent claims spaces conditioned on state probabilities. For the set of state probabilities Δ^{S-1}, suppose that the corresponding set $\{\preceq_{\boldsymbol{\pi}}\}$ exists and is representable by $U(\mathbf{c};\boldsymbol{\pi})$. We first give the representation result over each contingent claim space, where $\boldsymbol{\pi}$ is specified. Then we investigate the incremental axioms which are necessary and sufficient for the Expected Utility representation for each preference relation in the set $\{\preceq_{\boldsymbol{\pi}}\}$ to have the same NM index v, up to a positive affine transform, on each slice. We compare and contrast axioms in our risky setting with related axioms in the SEU setting.

3.1 Representation over Each Contingent Claim Space

In this subsection, we first consider the standard contingent claim setting where for each specific $\boldsymbol{\pi}$, $U(\mathbf{c};\boldsymbol{\pi})$ takes the state independent Expected Utility form as in (5). We provide the necessary and sufficient conditions for this to be the case.

Based on the SEU (Subjective Expected Utility) literature a natural candidate axiom for $U(\mathbf{c};\boldsymbol{\pi})$ to become a state independent Expected Utility is the following version of the Tradeoff Consistency axiom introduced by Wakker (1989).[3]

Axiom 1 *(Tradeoff Consistency) For each* $\boldsymbol{\pi} \in \Delta^{S-1}$, *if* $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$, $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$ *and* $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}} \mathbf{c}''_{-s'}x$, *then* $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}} \mathbf{c}''_{-s'}z$, *where* $\mathbf{c}_{-s}x$ *denotes the consumption vector* $\mathbf{c}$ *with consumption* c_s *in state* s *replaced by* x.

It follows from Wakker (1984, Lemmas 2.2 and 2.4) that Axiom 1 implies the Sure-Thing Principle

$$\mathbf{c}_{-s}x \preceq_{\boldsymbol{\pi}} \mathbf{c}'_{-s}x \Leftrightarrow \mathbf{c}_{-s}y \preceq_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y \tag{15}$$

and the Thomsen-Blaschke condition when $S = 2$,

$$(c_1, c_2) \sim_{\boldsymbol{\pi}} (c'_1, c'_2), (c_1, c'_2) \sim_{\boldsymbol{\pi}} (c'_1, c''_2), (c''_1, c'_2) \sim_{\boldsymbol{\pi}} (c_1, c''_2) \Rightarrow (c''_1, c_2) \sim_{\boldsymbol{\pi}} (c_1, c''_2). \tag{16}$$

Therefore Axiom 1 implies that the utility function is additively separable. However this axiom is not enough to ensure the existence of a state independent Expected Utility representation where the probabilities given exogenously as opposed to endogenously determined in the SEU formulation. To derive the representation result, we assume the following Risk Aversion axiom proposed by Werner (2005).[4]

Axiom 2 *(Risk Aversion) For each* $\boldsymbol{\pi} \in \Delta^{S-1}$ *and a given* $\mathbf{c} \in \mathbb{R}^S_+$,

$$\mathbf{c} \preceq_{\boldsymbol{\pi}} E_{\boldsymbol{\pi}}(\mathbf{c}), \tag{17}$$

where $E_{\boldsymbol{\pi}}(\mathbf{c})$ *denotes the* S*-vector* $\bar{\mathbf{c}}$ *for which* $\bar{c}_s = \sum_{i=1}^S \pi_i c_i$ *for each* s.

Then we have the following result.

Theorem 1 *For each* $\boldsymbol{\pi} \in \Delta^{S-1}$, $U(\mathbf{c}; \boldsymbol{\pi})$ *takes the following functional form*[5]

$$U(\mathbf{c}; \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^S \pi_s v_{\boldsymbol{\pi}}(c_s)\right), \tag{18}$$

[3] The SEU setting is considered in the seminal paper of Savage (1954) and further investigated in an extensive literature including the important papers of Anscombe and Aumann (1963) and Wakker (1989). For a more complete discussion of the SEU framework and associated axioms, see, for example, Wakker (1989), Nau (2011) and Karni (2013).

[4] The intuition for the Risk Aversion axiom is that a certain payoff is always preferred to the uncertain payoff with the same mean. The surprising part is that with the assumption of additive separability, this axiom ensures that probabilities enter into the utility function linearly.

[5] As noted in Section 2, $U(\mathbf{c}; \boldsymbol{\pi})$ is assumed to be a strictly increasing function. This can be guaranteed by the monotonicity of $v_{\boldsymbol{\pi}}$ and f in its second argument. Note that if $v_{\boldsymbol{\pi}}$ is strictly decreasing and f is also strictly decreasing in its second argument, $U(\mathbf{c}; \boldsymbol{\pi})$ is still a strictly increasing function. But this case can be also achieved by assuming both to be strictly increasing and thus it is ignored in Theorem 1. A similar argument holds as well for Theorems 2 - 4.

where f is an arbitrary function that can depend on $\boldsymbol{\pi}$ and is strictly increasing in its second argument and $v_{\boldsymbol{\pi}}$ is a strictly increasing and concave function, if and only if Axioms 1 and 2 hold.

Proof. See Werner (2005) for the proof. ■

It will be noted that each NM index $v_{\boldsymbol{\pi}}$ is allowed to depend on probabilities. This is consistent with the utility (1) discussed in Section 2, which takes the form of $U(\mathbf{c};\boldsymbol{\pi})$ in Theorem 1. Indeed it can readily be verified that (1) satisfies Axioms 1 and 2 for each given probability vector.

In the SEU setting, the Tradeoff Consistency axiom by itself is necessary and sufficient for a state independent Expected Utility representation. Why do we need to also assume Axiom 2 in our setting? To answer this question, note that in the SEU setting the state independent Expected Utility takes the form

$$U(\mathbf{c}) = \sum_{s=1}^{S} \omega_s v(c_s), \tag{19}$$

where $v(c_s)$ is a state-independent utility function, unique up to a positive affine transformation, and $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_S)$ is a uniquely determined probability vector.[6] However in our setting since probabilities are given exogenously and not endogenously determined, the Tradeoff Consistency axiom can not ensure the endogenously determined $\boldsymbol{\omega}$ matches the exogenously given $\boldsymbol{\pi}$. To see this more explicitly, consider the following two examples. The first one can be viewed as a variant of a Prospect Theory representation and the second one can be viewed a state dependent Expected Utility. Both examples satisfy Tradeoff Consistency.

Example 1 *Assume that*

$$U(\mathbf{c};\boldsymbol{\pi}) = \pi_1^2 v(c_1) + \pi_2^2 v(c_2) + \pi_3^2 v(c_3). \tag{20}$$

Note that this representation can be viewed as

$$U(\mathbf{c}) = \sum_{s=1}^{3} \omega_s v(c_s), \tag{21}$$

where

$$\omega_s = \frac{\pi_s^2}{\pi_1^2 + \pi_2^2 + \pi_3^2}. \tag{22}$$

[6] Since the SEU axioms imply the existence of a v and a $\boldsymbol{\omega}$, we use in (19) the notation $U(\mathbf{c})$ rather than $U(\mathbf{c};\boldsymbol{\omega})$ to reflect the fact that $\boldsymbol{\omega}$ should not be viewed as a parameter that can be changed like our exogenously given $\boldsymbol{\pi}$.

Clearly the utility (20) satisfies Tradeoff Consistency and is a state independent SEU function. Although the utility satisfies the state independence requirement of Axiom 2, it does not satisfy the requirement that the probabilities enter into the utility function linearly.

Example 2 *Assume that*

$$U(\mathbf{c};\boldsymbol{\pi}) = \pi_1 v\left(c_1\right) + 2\pi_2 v\left(c_2\right) + 3\pi_3 v\left(c_3\right). \tag{23}$$

Note that (23) can be rewritten as the state independent SEU

$$U(\mathbf{c}) = \sum_{s=1}^{3} \omega_s v(c_s), \tag{24}$$

where

$$\omega_1 = \frac{\pi_1}{\pi_1 + 2\pi_2 + 3\pi_3}, \omega_2 = \frac{2\pi_2}{\pi_1 + 2\pi_2 + 3\pi_3} \text{ and } \omega_3 = \frac{3\pi_3}{\pi_1 + 2\pi_2 + 3\pi_3} \tag{25}$$

and (23) satisfies Tradeoff Consistency. However in our setting, the probability vector $\boldsymbol{\pi}$ *is exogenous and fixed and cannot be transformed into* $\boldsymbol{\omega}$*. To see this implies that the utility (23) is not state independent, observe that it can be written as*

$$U(\mathbf{c};\boldsymbol{\pi}) = \sum_{s=1}^{3} \pi_s s v(c_s), \tag{26}$$

where the NM index in each state is given by

$$v_s(c_s) = s v(c_s), \tag{27}$$

which is clearly state dependent and violates our Axiom 2. Thus, the Tradeoff Consistency axiom in the SEU setting does not imply state independence in our setting, where probabilities are exogenous.

3.2 Representation over All Contingent Claim Spaces

Suppose rather than allowing the NM index v in Theorem 1 to vary as the state probabilities change, one wants to ensure that the set of preference relations $\{\preceq_{\boldsymbol{\pi}}\}$ are representable by a common state independent Expected Utility function across contingent claim slices as in Figure 1. As shown in eqn. (1), even if $U\left(\mathbf{c};\boldsymbol{\pi}\right)$ takes the state independent Expected Utility form in each contingent claim space, it may not be a state independent Expected Utility with respect to the set of preference relations $\{\preceq_{\boldsymbol{\pi}}\}$. Interestingly, this additional requirement can be achieved by simply modifying the Tradeoff Consistency axiom 1, which is applicable to our setting of multiple slices and multiple probability vectors.

Axiom 3 *(Modified Tradeoff Consistency) For each* $\boldsymbol{\pi} \in \Delta^{S-1}$, *if* $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$ *and* $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$ *then for any* $\boldsymbol{\pi}' \in \Delta^{S-1}$, *if* $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}x$, *then* $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}z$.

To provide some intuition for Axiom 3, assume $S = 2$ and consider the following consumption pairs

$$\mathbf{c} = \mathbf{c}'' = (c_1, 1), \mathbf{c}' = (c'_1, 0) \text{ and } \mathbf{c}''' = \left(c_1, \frac{1}{9}\right). \tag{28}$$

Consider two contingent claim slices corresponding to

$$\pi_1 = 0.5 \text{ and } \pi'_1 = 0.4 \tag{29}$$

and the consumption values

$$x = 1, y = 4, w = 16, z = 9. \tag{30}$$

Axiom 3 implies that if

$$(1,1) \sim_{\boldsymbol{\pi}} (4,0), (9,1) \sim_{\boldsymbol{\pi}} (16,0) \text{ and } (1,1) \sim_{\boldsymbol{\pi}'} \left(4, \frac{1}{9}\right), \tag{31}$$

then we must have

$$(9,1) \sim_{\boldsymbol{\pi}'} \left(16, \frac{1}{9}\right). \tag{32}$$

This chain of indifferent consumption pairs is shown in Figure 2(a) and (b) respectively, where we assume the Expected Utility representation

$$U(\mathbf{c}; \boldsymbol{\pi}) = \pi_1\sqrt{c_1} + \pi_2\sqrt{c_2}. \tag{33}$$

Axiom 3 is clearly satisfied.[7] In the SEU setting, since the probabilities are endogenously determined, one only considers the case with a fixed probability structure like Figure 2(a). Our contribution here is to assume Tradeoff Consistency holds where the probability structure changes as in Figure 2(b).

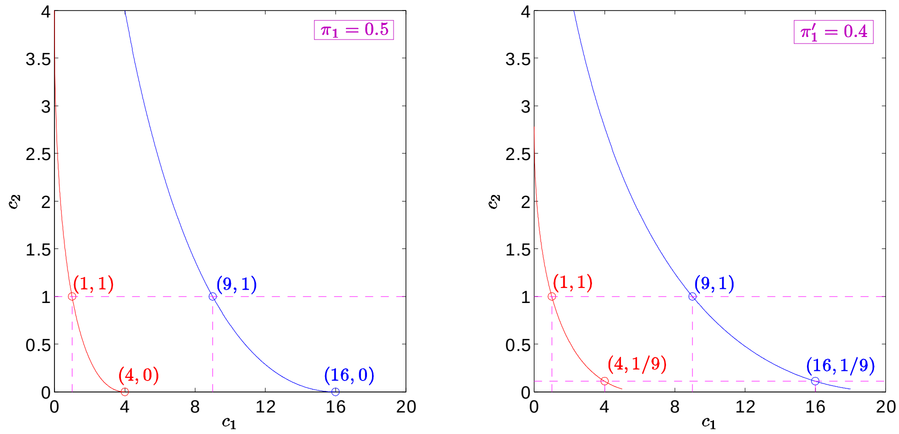
Then we have the following theorem.

Theorem 2 *For all* $\boldsymbol{\pi} \in \Delta^{S-1}$, $U(\mathbf{c}; \boldsymbol{\pi})$ *takes the following functional form*

$$U(\mathbf{c}; \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{34}$$

where f *is an arbitrary function that can depend on* $\boldsymbol{\pi}$ *and is strictly increasing in its second argument and* v *is a strictly increasing and concave function if and only if Axioms 2 and 3 hold.*

[7] It should be noted that Figure 2 is similar to Figure 4.5.1 in Kobberling and Wakker (2004).



(a) (b)

Figure 2:

Proof. Necessity is clear. Next prove sufficiency. Taking $\boldsymbol{\pi}' = \boldsymbol{\pi}$, it follows from Theorem 1 that Axioms 3 and 2 imply that

$$U(\mathbf{c}; \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right). \tag{35}$$

Suppose that for $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$, we have $\mathbf{c}_{-s}x \sim_{\boldsymbol{\pi}} \mathbf{c}'_{-s}y$ and $\mathbf{c}'_{-s}w \sim_{\boldsymbol{\pi}} \mathbf{c}_{-s}z$. Then

$$v_{\boldsymbol{\pi}}(x) - v_{\boldsymbol{\pi}}(y) = v_{\boldsymbol{\pi}}(z) - v_{\boldsymbol{\pi}}(w). \tag{36}$$

Similarly, $\mathbf{c}'''_{-s'}y \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}x$ and $\mathbf{c}'''_{-s'}w \sim_{\boldsymbol{\pi}'} \mathbf{c}''_{-s'}z$ imply that

$$v_{\boldsymbol{\pi}'}(x) - v_{\boldsymbol{\pi}'}(y) = v_{\boldsymbol{\pi}'}(z) - v_{\boldsymbol{\pi}'}(w). \tag{37}$$

Since Axiom 3 implies that eqns. (36) and (37) hold for any x, y, z, w, $v_{\boldsymbol{\pi}}$ and $v_{\boldsymbol{\pi}'}$ must be affinely equivalent. Therefore, for any $\boldsymbol{\pi} \neq \boldsymbol{\pi}' \in \Delta^{S-1}$, we must have

$$v_{\boldsymbol{\pi}} = a v_{\boldsymbol{\pi}'} + b, \tag{38}$$

where $a > 0$ and b are some constants. Since the NM index is defined up to an affine transformation, we can conclude that

$$U(\mathbf{c}; \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{39}$$

which completes the proof. ■

4 Preferences over Lotteries

In the previous section preferences were assumed to be defined over contingent consumption, and probabilities entered only as parameters. However suppose instead that a decision maker faces choices over different "risky prospects" or lotteries, which are defined as vectors $(c, \pi) \in R_+^S \times \Delta^{S-1}$. As described in Section 2, we assume a continuous, complete and transitive preference ordering over $P = R_+^S \times \Delta^{S-1}$ denoted by $\preceq_{\mathcal{P}}$. In this section, we consider what additional axioms beyond those in Theorem 2 are required to extend the state independent Expected Utility representation of $\{\preceq_{\boldsymbol{\pi}}\}$ to $\preceq_{\mathcal{P}}$. Maintaining Axioms 2 and 3, the following turns out to be necessary and sufficient.

Axiom 4 *(Certainty Uniqueness) For any certain consumption $\mathbf{c} = (c_1, c_2, ..., c_S)$, where $c_s = \overline{c}$ is a constant for each state s,*

$$(\mathbf{c}, \boldsymbol{\pi}) \sim_{\mathcal{P}} (\mathbf{c}, \boldsymbol{\pi}') \ \ \forall \boldsymbol{\pi}, \boldsymbol{\pi}' \in \Delta^{S-1}. \tag{40}$$

Axiom 4 assumes that the decision maker is indifferent between the same certain consumption vector on different contingent claim spaces parameterized by different probability vectors. In terms of Figure 1, this corresponds to being indifferent to the same $\mathbf{c}$ point along the 45° rays on the slices corresponding to $\boldsymbol{\pi}'$ and $\boldsymbol{\pi}''$. Based on Axiom 4, we assume throughout this section that for any certain consumption vectors $\mathbf{c} = (c_1, c_2, ..., c_S)$ and $\mathbf{c}' = (c_1', c_2', ..., c_S')$ where $c_s = \overline{c}$ and $c_s' = \overline{c}'$ $(\forall s \in \{1, 2, ..., S\})$ and for any $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \Delta^{S-1}$, $(\mathbf{c}, \boldsymbol{\pi}) \preceq_{\mathcal{P}} (\mathbf{c}', \boldsymbol{\pi}')$ if and only if $\overline{c} \leq \overline{c}'$.

Then we have the following theorem.

Theorem 3 *$U(\mathbf{c}, \boldsymbol{\pi})$ representing $\preceq_{\mathcal{P}}$ takes the following functional form*

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{41}$$

where f is a strictly increasing function independent of probabilities and v is a strictly increasing and concave function if and only if Axioms 2, 3 and 4 hold.

Proof. Necessity is obvious. Next we prove sufficiency. It follows from Theorem 2 that Axioms 2 and 3 are equivalent to a utility representation of the form

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v(c_s)\right), \tag{42}$$

where $v(c_s)$ is a strictly increasing and concave function. It follows from Axiom 4 that $\forall \mathbf{c} = (\overline{c}, \overline{c}, ..., \overline{c}) \in \mathbb{R}_+^S$ and $\forall \boldsymbol{\pi}, \boldsymbol{\pi}' \in \Delta^{S-1}$, we have

$$f(\boldsymbol{\pi}, v(\overline{c})) = f(\boldsymbol{\pi}', v(\overline{c})), \tag{43}$$

implying that $f(\cdot, \cdot)$ must be independent of probabilities. ∎

Comparing the representations (34) and (41) in Theorems 2 and 3, respectively, Axiom 4 is necessary and sufficient for the transformation f to be independent of $\boldsymbol{\pi}$. For example, the introduction of Axiom 4 rules out eqn. (1) in Section 2 as a possible representation of $\preceq_{\mathcal{P}}$. It should be stressed that the form of utility in Theorem 3 is not verifiable at the demand level since whether or not the transformation f depends on probabilities cannot be determined from the contingent claim demand functions.

It is natural to wonder whether it is enough to use the Tradeoff Consistency Axiom 1 instead of the modified version Axiom 3 together with Axioms 2 and 4 to obtain the desired result in Theorem 3. Unfortunately as the following example shows, this is not the case.

Example 3 *Assume that*

$$U(\mathbf{c}, \boldsymbol{\pi}) = f\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = \left(\pi_1 c_1^{\frac{\pi_1}{2}} + \pi_2 c_2^{\frac{\pi_1}{2}} + \pi_3 c_3^{\frac{\pi_1}{2}}\right)^{\frac{1}{\pi_1}}, \tag{44}$$

where

$$v_{\boldsymbol{\pi}}(c_s) = c_s^{\frac{\pi_1}{2}} \quad \text{and} \quad f(\boldsymbol{\pi}, x) = x^{\frac{1}{\pi_1}}. \tag{45}$$

If consumption in each of the states is the same, $c_s = \overline{c}$, then

$$U(\mathbf{c}, \boldsymbol{\pi}) = \left(\pi_1 \overline{c}^{\frac{\pi_1}{2}} + \pi_2 \overline{c}^{\frac{\pi_1}{2}} + \pi_3 \overline{c}^{\frac{\pi_1}{2}}\right)^{\frac{1}{\pi_1}} = \sqrt{\overline{c}}, \tag{46}$$

which is independent of probabilities and hence Axiom 4 holds. For each fixed probability $\boldsymbol{\pi}$, (44) is clearly a state independent Expected Utility. Therefore, Axioms 2 and 1 hold. But obviously (44) does not take the form of (41) in Theorem 3.

Assuming Axioms 2 and 1 hold, is it possible to replace Axiom 4 by another axiom which ensures that U takes the form in (41)? Before introducing a new axiom, we define some additional notation. For any $(\mathbf{c}, \boldsymbol{\pi})$, where $\boldsymbol{\pi} \in \Delta^{S-1}$, assuming $(\mathbf{c}, \boldsymbol{\pi})$ corresponds to the random variable X, the cumulative distribution function is

$$F_X(z) = \sum_{s=1}^{S} \pi_s I(c_s \leq z), \tag{47}$$

where

$$I\left(c_s \leq z\right)=\begin{cases}1 & \left(c_s \leq z\right) \\ 0 & \left(c_s>z\right)\end{cases}. \tag{48}$$

Axiom 5 *For any pair of random variables X and Y corresponding, respectively, to $(\mathbf{c}, \boldsymbol{\pi})$ and $(\mathbf{c}', \boldsymbol{\pi}')$, where $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \Delta^{S-1}$, if $F_X(z)=F_Y(z)$, then*

$$(\mathbf{c}, \boldsymbol{\pi}) \sim_{\mathcal{P}} (\mathbf{c}', \boldsymbol{\pi}'). \tag{49}$$

The intuition for this axiom is that for any pair of lotteries defined on different contingent claim spaces, if their respective cumulative distribution functions are the same, then the lotteries will be indifferent. This is consistent with both the NM index v and the transformation f being independent of $\boldsymbol{\pi}$. It is clear that Axiom 5 implies Axiom 4.

Remark 2 *Axiom 5 will be recognized to be similar to the probabilistic sophistication property introduced by Machina and Schmeidler (1992) in an SEU setting (also see Grant, Özsoy and Polak 2008). Because this property is based on subjective probabilities, it is necessary to introduce axiomatic structure to ensure that the endogenous probabilities satisfy probabilistic sophistication. However in the case of Axiom 5, the probabilities are given exogenously and the axiom can be directly assumed.*

We next show that Axiom 5 together with Axioms 2 and 1 are necessary and sufficient for $\preceq_{\mathcal{P}}$ to be representable by a state independent Expected Utility function where the NM index does not depend on probabilities in contrast to the case of Example 3.

Theorem 4 *When $S>2$, $U(\mathbf{c}, \boldsymbol{\pi})$ representing $\preceq_{\mathcal{P}}$ takes the following functional form*

$$U(\mathbf{c}, \boldsymbol{\pi})=f\left(\sum_{s=1}^{S} \pi_s v\left(c_s\right)\right), \tag{50}$$

where f is a strictly increasing function independent of probabilities and v is a strictly increasing and concave function if and only if Axioms 1, 2 and 5 hold.

Proof. Necessity is obvious. Next we prove sufficiency. It follows from Theorem 1 that Axioms 2 and 1 are equivalent to a utility representation of the form

$$U(\mathbf{c}, \boldsymbol{\pi})=g\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right). \tag{51}$$

Axiom 5 implies that

$$U(\mathbf{c},\boldsymbol{\pi}) = g\left(\boldsymbol{\pi}, v_{\boldsymbol{\pi}}(c)\right) \tag{52}$$

is independent of probabilities. Assume that

$$g_{\boldsymbol{\pi}}\left(v_{\boldsymbol{\pi}}(c)\right) = f\left(c\right), \tag{53}$$

where f is independent of probabilities. It follows that $\forall c$

$$g_{\boldsymbol{\pi}}\left(c\right) = g_{\boldsymbol{\pi}}\left(v_{\boldsymbol{\pi}} \circ v_{\boldsymbol{\pi}}^{-1}(c)\right) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(c\right), \tag{54}$$

implying that

$$g_{\boldsymbol{\pi}} = f \circ v_{\boldsymbol{\pi}}^{-1}. \tag{55}$$

If $c_1 \neq c_2 = c_3 = ... = c_S = \overline{c}$ then it follows from Axiom 5 that

$$U(\mathbf{c},\boldsymbol{\pi}) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(\left(1-\pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right) \tag{56}$$

is independent of π_s $(s > 1)$, or equivalently,

$$\frac{\partial v_{\boldsymbol{\pi}}^{-1}\left(\left(1-\pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right)}{\partial \pi_s} = 0 \quad \left(\forall s = 2,3,...,S\right). \tag{57}$$

Holding π_1 fixed, consider two different profiles of probabilities $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ with associated NM indices $v_{\boldsymbol{\pi}}$ and $v_{\boldsymbol{\pi}'}$. It follows from (57) that there exists a $\eta(c_1,\overline{c})$ such that

$$\eta(c_1,\overline{c}) = v_{\boldsymbol{\pi}}^{-1}\left(\left(1-\pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1)\right) \tag{58}$$

and

$$\eta(c_1,\overline{c}) = v_{\boldsymbol{\pi}'}^{-1}\left(\left(1-\pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1)\right), \tag{59}$$

implying that

$$v_{\boldsymbol{\pi}}\left(\eta(c_1,\overline{c})\right) = \left(1-\pi_1\right) v_{\boldsymbol{\pi}}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}}(c_1) \tag{60}$$

and

$$v_{\boldsymbol{\pi}'}\left(\eta(c_1,\overline{c})\right) = \left(1-\pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1). \tag{61}$$

Therefore,

$$\begin{aligned} v_{\boldsymbol{\pi}}\left(\eta(c_1,\overline{c})\right)) &= h(v_{\boldsymbol{\pi}'}\left(\eta(c_1,\overline{c})\right)) \\ &= \pi_1 h(v_{\boldsymbol{\pi}'}(c_1)) + (1-\pi_1) h(v_{\boldsymbol{\pi}'}(\overline{c})) \\ &= h\left(\left(1-\pi_1\right) v_{\boldsymbol{\pi}'}(\overline{c}) + \pi_1 v_{\boldsymbol{\pi}'}(c_1)\right). \end{aligned} \tag{62}$$

Since eqn. (62) holds for any given π_1, c_1 and $\overline{c}$, we can conclude that h is a linear function which can depend on π_s and π'_s $(s = 2,3,...,S)$. Since $\forall \boldsymbol{\pi},\boldsymbol{\pi}' \in \Delta^{S-1}$

with the same π_1, there always exists a linear function $h_{\boldsymbol{\pi},\boldsymbol{\pi}'}(\cdot)$ such that $v_{\boldsymbol{\pi}}(c) = h_{\boldsymbol{\pi},\boldsymbol{\pi}'}(v_{\boldsymbol{\pi}'}(c))$, we can conclude that

$$v_{\boldsymbol{\pi}}(c) = \kappa_1'(\pi_1, \pi_2, ..., \pi_{S-1})\, v_{\pi_1}(c) + \kappa_2'(\pi_1, \pi_2, ..., \pi_{S-1}) \quad (\forall c), \tag{63}$$

where κ_1' and κ_2' are some arbitrary coefficients. Assuming $c_2 \neq c_1 = c_3 = ... = c_S = \bar{c}$ and following the similar argument, we can also show that

$$v_{\boldsymbol{\pi}}(c) = \kappa_1''(\pi_1, \pi_2, ..., \pi_{S-1})\, v_{\pi_2}(c) + \kappa_2''(\pi_1, \pi_2, ..., \pi_{S-1}) \quad (\forall c), \tag{64}$$

where κ_1'' and κ_2'' are some arbitrary coefficients. Combining eqn. (63) with (64) yields

$$v_{\boldsymbol{\pi}}(c) = \kappa_1(\pi_1, \pi_2, ..., \pi_{S-1})\, v(c) + \kappa_2(\pi_1, \pi_2, ..., \pi_{S-1}) \quad (\forall c), \tag{65}$$

where κ_1 and κ_2 are some arbitrary coefficients and v is independent of probabilities. Therefore we have

$$U(\mathbf{c}, \boldsymbol{\pi}) = g\left(\boldsymbol{\pi}, \sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = f \circ v_{\boldsymbol{\pi}}^{-1}\left(\sum_{s=1}^{S} \pi_s v_{\boldsymbol{\pi}}(c_s)\right) = f\left(\sum_{s=1}^{S} \pi_s v(c_s)\right). \tag{66}$$

■

Remark 3 *We can compare the conditions in Theorems 3 and 4. For Theorem 3, although the Certainty Uniqueness axiom is easy to test, one needs to consider the modified Tradeoff Consistency axiom. That means one needs to verify whether the Tradeoff Consistency axiom holds on different contingent claim spaces as shown in Figure 2, which is complicated in general. For Theorem 4, one just needs to consider the traditional Tradeoff Consistency axiom and Axiom 5. In principle to conduct a laboratory test based on Theorem 4 is easier than based on Theorem 3. However, it should be noted that Theorem 4 only works for the* $S > 2$ *case since Axiom 5 converges to Axiom 4 when* $S = 2$*. Therefore, if there are only two states, one has no choice but use Theorem 3.*

Finally, it is natural to inquire into the relationship between Theorem 4 and the conventional Expected Utility representation result based on the Strong Independence axiom (e.g., Samuelson 1952 and Grandmont 1972). First let $\mathcal{F}$ denote the set of all cumulative distribution functions defined on the consumption space $(0, \infty)$. Assume preferences are defined over $\mathcal{F}$, which is a mixture space. Since $\mathcal{F}$ consists all possible distributions, it is not restricted to S states. However, it can be easily seen that (50) is also the Expected Utility representation over $\mathcal{F}$, if one restricts the number of the lottery states to be less than or equal to S. Indeed the Strong Independence axiom typically assumed for preferences over $\mathcal{F}$ holds for any mixture of lotteries where the maximum number of states of the lotteries is S. Therefore, the only difference between the set of risk prospects $\mathcal{P}$ assumed in this section and $\mathcal{F}$ is that for the former the number of the states are fixed at S and for $\mathcal{F}$, there is no restriction to the number of states.

5 Marschak-Machina Triangle

As noted above, although probability dependent transformations of state independent Expected Utility functions will not alter contingent claim demand behavior, they do change the consumer's preferences over lotteries. To see the implications of this, we consider in this section an example which utilizes the probability simplex proposed by Marschak (1950) and extended by Machina (1982) (often referred to as the MM (Marschak-Machina) triangle). It should be noted that in any given MM triangle, the payoff c_s $(s = 1, 2, 3)$ on each vertex is fixed and each point in the triangle corresponds to a different probability vector. We follow the convention of associating the largest, middle and smallest values of c_s with the northern, southeastern and southwestern vertices of the triangle, respectively.

Example 4 *Assume the following demand functions*

$$c_1 = \left(\frac{1}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_1}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_1}}} + \frac{1}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_2}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_2}}} \right) \frac{I}{2}, \tag{67}$$

$$c_2 = \left(\frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_1}}}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_1}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_1}}} + \frac{\left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_2}}}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_2}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_2}}} \right) \frac{I}{2} \tag{68}$$

and

$$c_3 = \left(\frac{\left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_1}}}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_1}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_1}}} + \frac{\left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_2}}}{p_1 + p_2 \left(\frac{\pi_2 p_1}{\pi_1 p_2}\right)^{\frac{1}{1+\delta_2}} + p_3 \left(\frac{\pi_3 p_1}{\pi_1 p_3}\right)^{\frac{1}{1+\delta_2}}} \right) \frac{I}{2}, \tag{69}$$

where $\delta_1 \neq \delta_2 > -1$. These demands are well-behaved in the sense that the associated Slutsky matrix is negative semidefinite and symmetric, implying that (67) - (69) are consistent with the maximization of a quasiconcave utility function subject to the standard contingent claim budget constraint. They exhibit normal good behavior and since each demand is linear in income, the underlying preferences are homothetic. However, the system of equations (67) - (69) does not satisfy the *demand tests in Kubler, Selden and Wei (2014) for preferences to be represented by an Expected Utility function and hence the corresponding utility* $U(\mathbf{x};\boldsymbol{\pi})$ *does not take the Expected Utility form in Theorem 2. Then what is the form of the utility generating these well-behaved demands? Denoting*

$$q_1 = \frac{p_1}{p_3}, q_2 = \frac{p_2}{p_3} \text{ and } m = \frac{I}{p_3} \tag{70}$$

and assuming $\delta_1 = -\frac{1}{2}$ *and* $\delta_2 = 0$*, we obtain the following demands*

$$c_1 = \left(\frac{\pi_1^2}{\pi_1^2 q_1 + \frac{\pi_2^2 q_1^2}{q_2} + \pi_3^2 q_1^2} + \frac{\pi_1}{q_1} \right) \frac{m}{2}, \tag{71}$$

$$c_2 = \left(\frac{\frac{\pi_2^2 q_1}{q_2^2}}{\pi_1^2 + \frac{\pi_2^2 q_1}{q_2} + \pi_3^2 q_1} + \frac{\pi_2}{q_2} \right) \frac{m}{2} \tag{72}$$

and

$$c_3 = \left(\frac{\pi_3^2 q_1}{\pi_1^2 + \frac{\pi_2^2 q_1}{q_2} + \pi_3^2 q_1} + \pi_3 \right) \frac{m}{2}, \tag{73}$$

and the indirect utility function[8]

$$V\left(q_1, q_2, m\right) = \frac{m\sqrt{q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^2 + \left(\frac{\pi_3 q_1}{\pi_1}\right)^2}}{q_1^{1+\frac{1}{2}\pi_1}}. \tag{74}$$

(See the Appendix for the general derivation in terms of δ_1 *and* δ_2*.) To generate an indifference curve in the MM triangle, one can fix a set of values corresponding to* (V, c_1, c_2, c_3) *and then solve for the set* (q_1, q_2, m, π_3) *when changing* π_1*. In Figure 3, we fix* $(c_1, c_2, c_3) = (1, 2, 3)$ *and draw two indifference curves corresponding to* $V = 6$ *(green curve) and* $V = 7$ *(red curve), respectively. The shape of the indifference curves clearly do not take the linear form associated with Expected Utility preferences. For general preferences over lotteries, it is standard to assume that preferences satisfy first-order stochastic dominance property. This condition requires that for any two random variables* X *and* Y*, if* $F_X(z) \leq F_Y(z)$*, then* X *is always preferred to* Y*. It is clear that first-order stochastic dominance implies our Axiom 4 (and hence Axiom 5). In the MM triangle, first-order stochastic dominance condition suggests the indifference cannot bend back as in Figure 3. (Graphically, if* X *first-order stochastic dominates* Y*,* X *lies to the northwest of* Y *in the MM triangle.) Therefore for preferences corresponding to the indirect*

[8] Despite the relatively simple form of the demand functions, it is not possible to obtain an analytical expression for $U(\mathbf{x};\boldsymbol{\pi})$.

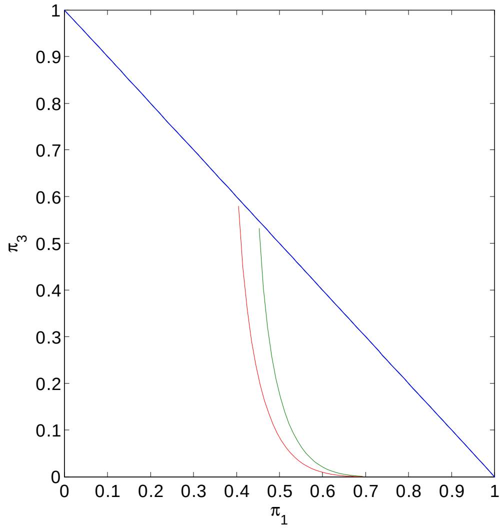


Figure 3:

utility eqn. (74), first-order stochastic dominance condition is violated.[9] *To see that the utility corresponding to (74) also violates Axioms 4 and 5, observe that* $c_1 = c_2 = c_3 = \bar{c}$ *implies that*

$$q_1 = \frac{\pi_1}{\pi_3},\ q_2 = \frac{\pi_2}{\pi_3} \text{ and } m = \frac{p_1\bar{c} + p_2\bar{c} + p_3\bar{c}}{p_3} = \frac{\bar{c}}{\pi_3}, \tag{75}$$

and[10]

$$U|_{c_1=c_2=c_3=\bar{c}} = \frac{\frac{\bar{c}}{\pi_3}\sqrt{\frac{\pi_1}{\pi_3} + \frac{\pi_2}{\pi_3} + 1}}{\left(\frac{\pi_1}{\pi_3}\right)^{1+\frac{1}{2}\pi_1}} = \frac{\bar{c}}{\pi_1^{1+\frac{1}{2}\pi_1}\pi_3^{\frac{1}{2}-\frac{1}{2}\pi_1}}. \tag{76}$$

Clearly the value of $U|_{c_1=c_2=c_3=\bar{c}}$ *depends on the probabilities. It is natural to wonder whether there exists some transformation* $f_{\boldsymbol{\pi}}$ *to make the representation satisfy Axiom 4. And if this is possible, how will the transformation affect the shape of the indifference curves in the MM triangle? In order to make* $U|_{c_1=c_2=c_3=\bar{c}}$ *independent of the probabilities and satisfy Axiom 4, one can apply the transformation*

$$f_{\boldsymbol{\pi}} \circ U = \pi_1^{1+\frac{1}{2}\pi_1}\pi_3^{\frac{1}{2}-\frac{1}{2}\pi_1}U. \tag{77}$$

[9] See Camerer and Ho (1994) for a more detailed discussion and illustration of indifference curve properties in the MM triangle.

[10] Noting that the direct utility function U and the indirect utility function V only differ in arguments, we can use the form of the indirect utility function to get the direct utility function when $c_1 = c_2 = c_3$.

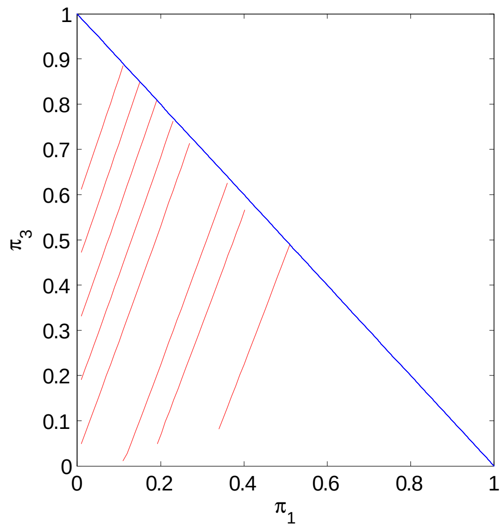


Figure 4:

Using $f_{\boldsymbol{\pi}}$, the indirect utility function defined in (74) can be transformed into

$$V = \pi_1^{1+\frac{1}{2}\pi_1} \pi_3^{\frac{1}{2}-\frac{1}{2}\pi_1} \frac{m\sqrt{q_1 + q_2\left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^2 + \left(\frac{\pi_3 q_1}{\pi_1}\right)^2}}{q_1^{1+\frac{1}{2}\pi_1}}. \tag{78}$$

Fixing $(c_1, c_2, c_3) = (1, 2, 3)$, we plot the indifference curves corresponding to (78) in Figure 4. It is clear that the indifference curves are very close to parallel lines corresponding to the Expected Utility case and hence first-order stochastic dominance property appears to hold. Thus corresponding to the demand system (67)-(69), it possible to have the two very different sets of indifference curves plotted in Figures 3 and 4.

It is clear from the discussion of the example in this section that preference properties such as first order stochastic dominance relating to the shape of the indifference curves in the MM triangle fail to be distinguishable at the corresponding contingent claim demand level. In fact, the set of lotteries in the MM triangle can be viewed as orthogonal to the set of lotteries in the contingent claims spaces parameterized by $\boldsymbol{\pi}$. In the contingent claim space, since the probabilities are fixed, any transformation based on probabilities $f_{\boldsymbol{\pi}}$ will not change the shape of the indifference curves. Similarly, in the MM triangle, since the payoffs are fixed, a transformation based on consumption values denoted by $f_{\mathbf{c}}$ will not change the shape of the indifference curves. But it is obvious that both the transformation $f_{\boldsymbol{\pi}}$ and $f_{\mathbf{c}}$ will affect general lottery comparisons in $\mathcal{P}$. The relationship between preferences defined on these three spaces and the corresponding transformations

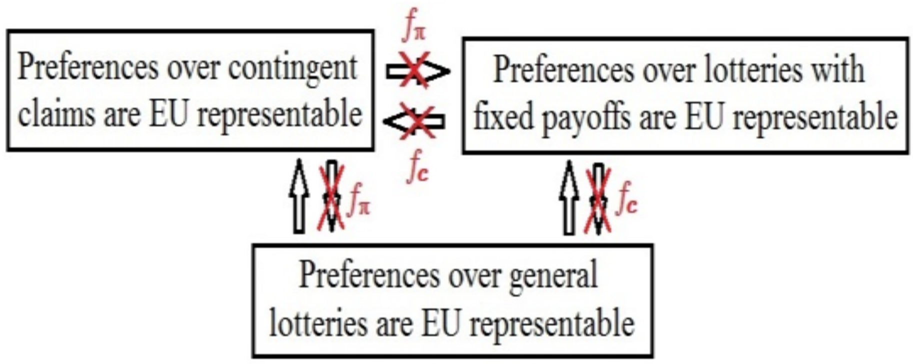


Figure 5:

is summarized in Figure 5. It follows, as suggested by the Figure, that the existence of an Expected Utility representation for lotteries defined in the contingent claim space cannot ensure an Expected Utility representation over lotteries corresponding to the MM triangle and vice versa. However, if the preferences over lotteries in $\mathcal{P}$ are Expected Utility representable, taking the form in eqn. (50), then the ordering will be Expected Utility representable for lotteries defined in the contingent claim space and the MM triangle.

Appendix

A Indirect Utility Function for General δ_1 and δ_2

In this appendix, we derive the indirect utility function that generates the demand functions in eqns. (67)-(69). Denoting

$$q_1 = \frac{p_1}{p_3}, q_2 = \frac{p_2}{p_3} \text{ and } m = \frac{I}{p_3} \tag{79}$$

and letting $\mu(p)$ denote the income compensation function, we have

$$\frac{\partial \ln \mu}{\partial q_1} = \left(\frac{1}{q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}}} + \frac{1}{q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_2}}} \right) \frac{1}{2} \tag{80}$$

and

$$\frac{\partial \ln \mu}{\partial q_2} = \left(\frac{\left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}}}{q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}}} + \frac{\left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}}}{q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_2}}} \right) \frac{1}{2}. \tag{81}$$

It can be verified that

$$\frac{\partial^2 \ln \mu}{\partial q_1 \partial q_2} = \frac{\partial^2 \ln \mu}{\partial q_2 \partial q_1}. \tag{82}$$

Therefore the equation system (80)-(81) has a unique solution and we can simply integrate (80) to obtain the solution. If $\delta_1 \delta_2 \neq 0$, integrating (80) yields

$$\begin{aligned}
\ln \mu &= \frac{\delta_2 (1+\delta_1) \ln \left(q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}} \right)}{2\delta_1 \delta_2} \\
&\quad + \frac{\delta_1 (1+\delta_2) \ln \left(q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_2}} \right)}{2\delta_1 \delta_2} \\
&\quad - \frac{(\delta_1+\delta_2) \ln q_1}{2\delta_1 \delta_2} + C.
\end{aligned} \tag{83}$$

Therefore, we have

$$\begin{aligned}
\mu &= K \left(q_1^{\frac{\delta_1}{1+\delta_1}} + q_2 \left(\frac{\pi_2}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3}{\pi_1}\right)^{\frac{1}{1+\delta_1}} \right)^{\frac{1+\delta_1}{2\delta_1}} \\
&\quad \times \left(q_1^{\frac{\delta_2}{1+\delta_2}} + q_2 \left(\frac{\pi_2}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}} + \left(\frac{\pi_3}{\pi_1}\right)^{\frac{1}{1+\delta_2}} \right)^{\frac{1+\delta_2}{2\delta_2}}.
\end{aligned} \tag{84}$$

The indirect utility function is given by

$$\begin{aligned}
V(q_1, q_2, m) &= \frac{m}{\left(q_1^{\frac{\delta_1}{1+\delta_1}} + q_2 \left(\frac{\pi_2}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3}{\pi_1}\right)^{\frac{1}{1+\delta_1}} \right)^{\frac{1+\delta_1}{2\delta_1}}} \\
&\quad \times \frac{1}{\left(q_1^{\frac{\delta_2}{1+\delta_2}} + q_2 \left(\frac{\pi_2}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_2}} + \left(\frac{\pi_3}{\pi_1}\right)^{\frac{1}{1+\delta_2}} \right)^{\frac{1+\delta_2}{2\delta_2}}}.
\end{aligned} \tag{85}$$

If $\delta_1 \delta_2 = 0$, since we consider the case when $\delta_1 \neq \delta_2$, without loss of generality, we assume that $\delta_2 = 0$ and then

$$\ln \mu = \frac{\pi_1 \delta_1 \ln q_1 + (1+\delta_1) \ln \left(q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}} \right) - \ln q_1}{2\delta_1} + C. \tag{86}$$

Therefore, we have

$$\mu = K \left(q_1 + q_2 \left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}} \right)^{\frac{1+\delta_1}{2\delta_1}} q_1^{\frac{\pi_1 \delta_1 - 1}{2\delta_1}}. \tag{87}$$

The indirect utility function is given by

$$V\left(q_1, q_2, m\right) = \frac{m}{\left(q_1 + q_2\left(\frac{\pi_2 q_1}{\pi_1 q_2}\right)^{\frac{1}{1+\delta_1}} + \left(\frac{\pi_3 q_1}{\pi_1}\right)^{\frac{1}{1+\delta_1}}\right)^{\frac{1+\delta_1}{2\delta_1}} q_1^{\frac{\pi_1\delta_1 - 1}{2\delta_1}}}. \tag{88}$$

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